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Kreiss symmetrizer and boundary conditions for the Euler–Korteweg system in a half space

Corentin Audiard

Institut Camille Jordan, Université Claude Bernard Lyon 1, 43 boulevard du 11 novembre 1918, 69622 Villeurbanne cedex, France

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ABSTRACT

The Euler–Korteweg system is a third order, dispersive system of PDEs, obtained from the standard Euler equations for compressible fluids by adding the so-called Korteweg stress tensor – encoding capillarity effects. Various results of well-posedness have been obtained recently for the Cauchy problem associated with the Euler–Korteweg system in the whole space. As to mixed problems, with initial and boundary value data, they are still mostly open. Here the linearized Euler–Korteweg system is studied in a half space by the use of normal mode analysis, which yields a generalized Kreiss–Lopatinskiĭ condition that must be satisfied by the boundary conditions for the boundary value problem to be well-posed.

Conversely, under the uniform Kreiss–Lopatinskiĭ condition, generalized Kreiss symmetrizers are constructed in one space dimension for an extended system originally introduced for the Cauchy problem, which displays crucial quasi-homogeneity properties. A priori estimates without loss of derivatives are thus derived, and finally the well-posedness of the mixed problem is obtained by combining the estimates for the pure boundary value problem and trace results for solutions of the pure Cauchy problem.

1. Introduction

The Euler–Korteweg system consists of the Euler equations with an additional “stress” term:

$$\begin{cases} \partial_t \rho + \operatorname{div}(\rho u) = 0, \\ \partial_t u + (u\cdot\nabla)u + \nabla g(\rho) = \nabla\left(K(\rho)\Delta\rho + \dfrac{1}{2}K'(\rho)|\nabla\rho|^2\right), \end{cases} \tag{1}$$

E-mail address: audiard@math.univ-lyon1.fr.

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where ρ represents the density, $u \in \mathbb{R}^d$ the velocity of a compressible fluid, $g : \rho \to g(\rho)$ (chemical potential) and $K : \rho \to K(\rho)$ (capillarity) are given smooth functions. The additional Korteweg tensor is intended to take into account capillarity effects. It is named after the work of Korteweg [7] in the XIXth century, see [13] for the derivation of the modern form of the equations.

This is a quasilinear third order system and it is known to admit local in time smooth solutions [1] and global smooth solutions such as traveling waves [2] in the whole space $\mathbb{R}^d$. A key point for the Cauchy problem analysis is the use of an additional dependent variable $v = \nabla\zeta$, where $\nabla\zeta = \sqrt{\frac{K'(\rho)}{\rho}}\nabla\rho$: the extended system in the variables (ζ, u, v) takes the form of a transport equation with source term for ζ coupled with a degenerate Schrödinger equation for $z = u + iv$. A priori estimates without loss of derivatives can then be obtained by integration by parts, suitable choices of weight functions and commutator estimates.

In this paper, we are concerned with initial boundary value problems (IBVP) for (1), which are more relevant from a physical point of view. As usual, they are more complicated to deal with than the Cauchy problem, in particular because of boundary terms in the integrations by parts. For dispersive PDEs in general, the IBVP has been addressed only recently (see [5]). For (1) in particular, it is mostly open.

As a first step, we consider the initial boundary value problem in a half space. Our aim is twofold:

(1) To characterize boundary conditions that are both physically reasonable and likely to yield suitable a priori estimates (possibly with no loss of derivatives);
(2) To actually prove a priori estimates for those boundary conditions, at least for the linearized problem with constant coefficients.

The analysis of a constant coefficient linear problem in the half space $\{x_d > 0\}$ is classically tackled by Fourier–Laplace transform, Laplace in time and Fourier in directions of the hyperplane $\{x_d = 0\}$, which transforms the PDE system into an ODE of the form $\partial_{x_d}\widehat{U} + G\widehat{U} = \widehat{U}_0$, where the matrix G depends on $\tau \in \mathbb{C}$, dual variable to t, and $\eta \in \mathbb{R}^{d-1}$, dual variable to $y = (x_1, \dots, x_{d-1})$, and $\widehat{U_0}$ is the Fourier transform of the initial condition for the unknown U.

Our approach is inspired by the strategy of Métivier and Zumbrun [11] for hyperbolic–parabolic IBVP, which consists in constructing quasi-homogeneous generalized Kreiss symmetrizers to obtain a priori estimates. However, we have to cope with two notable difficulties: the homogeneity properties of the system are definitely worse, due to dispersive third order terms, and some eigenvalues of G are purely imaginary when $\mathrm{Re}(\tau) = 0$ (as in hyperbolic IBVPs).

The first one is dealt with by working on the extended system rather than the initial one, since it appears to have better homogeneity properties. If $d \geqslant 2$, it turns out that the boundary value problem is characteristic, this is one of the reasons why we shall restrict the analysis to the one-dimensional case ($d = 1$). The second difficulty requires to study carefully the asymptotic behavior of G's eigenvalues. It turns out that in dimension 1 the eigenvalues of G split into two of positive real part and two of negative real part, which remain well separated when $\mathrm{Re}(\tau) \to 0$, at least if $|\tau|$ is large enough, which is the most relevant case. This will allow us to construct a Kreiss symmetrizer under a generalized Kreiss–Lopatinskiĭ condition.

The paper is organized as follows: the Kreiss–Lopatinskiĭ condition will be derived in Section 2 and we give an asymptotic development of G's eigenvalues. In Section 3, we introduce the extended system, exhibit physically reasonable boundary conditions that do satisfy the Kreiss–Lopatinskiĭ condition, and the actual construction of symmetrizers is done. The subsequent derivation of a priori estimates is followed in Section 4 by existence and uniqueness results.

2. Kreiss–Lopatinskiĭ condition for the linearized Euler–Korteweg system

We linearize the Euler–Korteweg equations (1) about a constant state $(\underline{\rho}, \underline{u})$, supposed to be thermodynamically stable which means $g'(\underline{\rho}) > 0$. In what follows, we use the simplified notations $\underline{g'} = g'(\underline{\rho})$, $\underline{K} = K(\underline{\rho})$.

The linearized system reads

$$\begin{cases} \partial_t \rho + \underline{u} \cdot \nabla \rho + \underline{\rho}\, \operatorname{div} u = 0, \\ \partial_t u + \underline{g}' \nabla \rho - \underline{K} \nabla \Delta \rho + \underline{u} \cdot \nabla u = 0. \end{cases} \tag{2}$$

For $\underline{\rho} > 0$, $\underline{g}' > 0$, $\underline{K} > 0$, the operator $(\rho, u) \mapsto (-u \cdot \nabla \rho - \underline{\rho}\, \operatorname{div} u, -g' \nabla \rho + \underline{K} \nabla \Delta \rho - \underline{u} \nabla u)$ is the infinitesimal generator of a C^0 semi-group of contractions on $H^1(\mathbb{R}^d) \times L^2(\mathbb{R}^2; \mathbb{R}^2)$ (see [2], Section 3.1). So the Cauchy problem is rather well understood.

To deal with the IBVP in the half space $\{x_d > 0\}$, we apply a Fourier transform in $(x_1, \dots, x_{d-1})$ and a Laplace transform in time, under which (2) becomes

$$\begin{cases} \tau \widehat{\rho} + \underline{v} \cdot i\eta \widehat{\rho} + \underline{w} \partial_d \widehat{\rho} + \underline{\rho}(i\eta \cdot \widehat{v} + \partial_d \widehat{w}) = \mathcal{F}(\rho_0), \\ \tau \widehat{v} + \underline{g}' i\eta \widehat{\rho} - \underline{K} i\eta \big(-|\eta|^2 + \partial_d^2\big) \widehat{\rho} + \underline{v} \cdot \eta \widehat{v} + \underline{w} \partial_d \widehat{v} = \mathcal{F}(v_0), \\ \tau \widehat{w} + \underline{g}' \partial_d \widehat{\rho} - \underline{K} \partial_d \big(-|\eta|^2 + \partial_d^2\big) \widehat{\rho} + \underline{v} \cdot \eta \widehat{v} + \underline{w} \partial_d \widehat{w} = \mathcal{F}(w_0), \end{cases} \tag{3}$$

where we have decomposed $\widehat{u} = (\widehat{v}, \widehat{w})$, $\widehat{v} \in \mathbb{C}^{d-1}$, $\widehat{w} \in \mathbb{C}$, $\operatorname{Re}(\tau) > 0$, $\eta \in \mathbb{R}^{d-1}$ and $\mathcal{F}$ is the Fourier transform on the $d-1$ first coordinates.

The equations in (3) can be written in the matrical form:

$$B \partial_d \widehat{U} = A(\tau, \eta) \widehat{U} + f, \tag{4}$$

where $\widehat{U} = (\widehat{\rho}, \partial_d \widehat{\rho}, \partial_d^2 \widehat{\rho}, i\widehat{v}, \widehat{w})$ and $f = (0, 0, -\mathcal{F}(w_0), i\mathcal{F}(v_0), \mathcal{F}(\rho_0))$. Up to a galilean transformation in the direction $x_d = 0$, we can choose $\underline{v} = 0$. Then the matrices reduce to:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & \underline{K} & 0 & -\underline{w} \\ 0 & 0 & 0 & \underline{w} & 0 \\ 0 & 0 & 0 & 0 & \underline{\rho} \end{pmatrix},$$

$$A(\tau, \eta) = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & \underline{g}' + \underline{K}|\eta|^2 & 0 & 0 & \tau \\ (\underline{g}' + \underline{K}|\eta|^2)\eta & 0 & -\underline{K}\eta & -\tau I_{d-1} & 0 \\ -\tau & -\underline{w} & 0 & -\rho\eta^t & 0 \end{pmatrix}.$$

If $\underline{w}\underline{\rho}\underline{K} \neq 0$, B is invertible, and the system (4) can be put in the form $\partial_d \widehat{U} = B^{-1} A \widehat{U} + B^{-1} f$. The characteristic polynomial reads

$$\begin{aligned} P(\omega; \tau, \eta) &:= \det\big(B^{-1} A(\tau, \eta) - \omega\big) \\ &= (\tau + \underline{w}\omega)^{d-1} \big((\tau + \underline{w}\omega)^2 - \underline{\rho}\big(\underline{g}' - \underline{K}\big(\omega^2 - \eta^2\big)\big)\big(\omega^2 - \eta^2\big)\big). \end{aligned} \tag{5}$$

If $d = 1$, we just have to set $\eta = 0$, and we find the reduced polynomial for the one-dimensional version of (4):

$$P(\omega; \tau) = (\tau + \underline{w}\omega)^2 - \underline{\rho}\big(\underline{g}' - \underline{K}\omega^2\big)\omega^2. \tag{6}$$

Proposition 1. *If* $\mathrm{Re}(\tau) > 0$, *the matrix* $B^{-1}A$ *is hyperbolic, i.e. it has no purely imaginary eigenvalue.*

Proof. Obviously, the root $\omega_0 = \frac{-\tau}{\underline{w}}$ associated to the factor $(\tau + \underline{w}\omega)^{d-1}$ in (5) is not purely imaginary if $\mathrm{Re}(\tau) > 0$.

Assume that $\omega = i\nu \in i\mathbb{R}$ is a root of the other factor in P, i.e.

$$(\tau + \underline{w}i\nu)^2 + \rho\big(\underline{g'} + \underline{K}\big(\nu^2 + \eta^2\big)\big)\big(\nu^2 + \eta^2\big) = 0. \tag{7}$$

Since $\rho(\underline{g'} + \underline{K}(\nu^2 + \eta^2))(\nu^2 + \eta^2) \in \mathbb{R}^+$, $\tau + i\underline{w}\nu$ must be purely imaginary and thus also τ has to be purely imaginary. $\square$

We assume here that boundary conditions $FU = \varphi(t, y)$ (where $y = (x_1, \ldots, x_{d-1}) \in \mathbb{R}^{d-1}$) are prescribed, with F a linear operator to be specified later.

We now work in a heuristic way to derive necessary algebraic properties of F for the IBVP to be well-posed. Let us consider (τ, η) as parameters of the BVP:

$$\begin{cases} \partial_{x_d}\widehat{U} = B^{-1}A\widehat{U} + f, & x_d > 0, \\ F\widehat{U} = \widehat{\varphi}, & x_d = 0. \end{cases} \tag{8}$$

If we want (8) to have just one solution $\widehat{U} \in L^2(\mathbb{R}^+)$, the boundary data must belong to the stable subspace E^- of $B^{-1}A$ (sum of the generalized eigenspaces corresponding to the eigenvalues of negative real part). So the existence and uniqueness of an L^2 solution amounts to the unique solvability of the algebraic system

$$\begin{cases} F\widehat{U}(0) = \widehat{\varphi}, \\ \widehat{U}(0) \in E^-. \end{cases}$$

Thus F must be an isomorphism $E^- \to \mathrm{Im}(F)$: this is the Lopatinskiĭ condition (for more details, see the introduction in [11]), which requires at least that E^- have a constant dimension on $\mathrm{Re}(\tau) > 0$, $\eta \in \mathbb{R}^{d-1}$ (the number of boundary conditions).

We investigate now this last condition. In order to simplify the notations, we will use a rescaled version of the polynomial P. We consider the nondimensional quantities

$$\widetilde{\tau} = \frac{\sqrt{\underline{\rho K}}}{\underline{c}^2}\tau, \qquad \widetilde{\omega} = \frac{\sqrt{\underline{\rho K}}}{\underline{c}}\omega, \qquad \widetilde{\eta} = \frac{\sqrt{\underline{\rho K}}}{\underline{c}}\eta, \qquad \underline{M} = \frac{\underline{w}}{\underline{c}} \quad \text{(the Mach number)}, \tag{9}$$

with $\underline{c} = \sqrt{\underline{g'}\,\underline{\rho}}$ the sound speed, and define

$$\widetilde{P}(\widetilde{\omega}; \widetilde{\tau}, \widetilde{\eta}) = \big((\widetilde{\tau} + \underline{M}\widetilde{\omega})^2 - \big(\widetilde{\omega}^2 - \widetilde{\eta}^2\big)\big(1 - \big(\widetilde{\omega}^2 - \widetilde{\eta}^2\big)\big)\big) \times (\widetilde{\tau} + \underline{M}\widetilde{\omega})^{d-1}. \tag{10}$$

It is easily checked that $P(\omega; \tau, \eta) = 0$ is equivalent to $\widetilde{P}(\widetilde{\omega}; \widetilde{\tau}, \widetilde{\eta}) = 0$.

In what follows, we omit the tildes for simplicity.

The roots of P consist of (obviously) $-\tau/\underline{M}$, of multiplicity $d - 1$, and those of $Q(\omega, \tau, \eta) = (\tau + \underline{M}\omega)^2 - (\omega^2 - \eta^2)(1 - (\omega^2 - \eta^2))$.

Lemma 1. *For* $\mathrm{Re}(\tau) > 0$, $\eta \in \mathbb{R}^{d-1}$, *there are two roots of* Q (*counted with multiplicity*) *in* $\{\mathrm{Re}(\omega) > 0\}$, *and two in* $\{\mathrm{Re}(\omega) < 0\}$.

Proof. We already know from Proposition 1 that $Q(\cdot;\tau,\eta)$ admits no purely imaginary root for $\operatorname{Re}(\tau)>0$. So it suffices by a connectedness argument to prove the result for some arbitrary (τ,η). We are going to study the case $(\tau,\eta)=(\tau,0)$, $\tau\in\mathbb{R}^{+*}$, $\tau\gg 1$. The equation $P(\omega,\tau,\eta)=0$ becomes

$$(\tau+\underline{M}\omega)^2-\omega^2\bigl(1-\omega^2\bigr)=0. \tag{11}$$

Dividing by τ^2, we get:

$$(1+\underline{M}\varepsilon\widehat{\omega})^2-\widehat{\omega}^2\bigl(\varepsilon^2-\widehat{\omega}^2\bigr)=0, \tag{12}$$

where $\widehat{\omega}:=\varepsilon\omega$, $\varepsilon=\frac{1}{\sqrt{\tau}}$.

For $\varepsilon=0$, Eq. (12) reduces to $1+\widehat{\omega}^4=0$, whose roots are $\pm\frac{1\pm i}{\sqrt{2}}$. Obviously, they are distinct, two have a positive real part and two a negative real part.

We conclude by continuity of the roots of (12) (see for example Kato [6], p. 107) that (11) has exactly two solutions of positive real parts and two solutions of negative reals part for all (τ,η) in the connected set $\{(\tau,\eta)\in\mathbb{C}\times\mathbb{R}^{d-1}:\ \operatorname{Re}(\tau)>0\}$. $\square$

Remark. This lemma indicates that the stable subspace of $B^{-1}A$ has a constant dimension, depending on the sign of $\underline{M}$ (or equivalently the sign of $\underline{w}$). Therefore, there should be 2 boundary conditions if $\underline{M}<0$, and $d+1$ if $\underline{M}>0$. That this number depends on $\underline{w}$ should not be surprising, since the first equation of (2) can be seen as an advection equation with speed $\underline{w}$ transversally to the boundary $\{x_d=0\}$.

In general, boundary conditions as in (8) have no reason to be 'dissipative', in the sense that a priori estimates cannot be obtained by direct energy methods. Symmetrizers are useful tools that were originally introduced by Kreiss [8] to cope with this problem for (homogeneous) hyperbolic (in the sense of PDEs) IBVPs.

Definition 1. A (generalized) symmetrizer S for (8) is a $d\times d$ self adjoint operator, depending smoothly on (τ,η) for $\gamma:=\operatorname{Re}(\tau)\geqslant 0$, uniformly bounded and satisfying:

$$\exists\alpha(\tau,\eta)>0:\quad S(\tau,\eta)B^{-1}A(\tau,\eta)\geqslant\gamma\alpha(\tau,\eta)I_d, \tag{13}$$

$$\exists C(\tau,\eta)>0,\ \exists\beta(\tau,\eta)>0:\quad S(\tau,\eta)\geqslant\beta(\tau,\eta)I_d-C(\tau,\eta)F^*F. \tag{14}$$

We can see the main interest of a symmetrizer in the following standard property:

Proposition 2. *If there is a symmetrizer S satisfying* (13), (14), *any smooth solution of* (8) *satisfies*:

$$\alpha(\tau,\eta)\frac{\gamma}{2}\|\widehat{U}\|^2+\beta(\tau,\eta)\bigl|\widehat{U}(0)\bigr|^2\leqslant C(\tau,\eta)|\widehat{\varphi}|^2+\frac{\|S\|_\infty^2}{2\gamma\alpha(\tau,\eta)}\|f\|^2, \tag{15}$$

where $|\cdot|$ is the usual euclidean norm in $\mathbb{C}^{2d+3}$, and $\|\cdot\|$ is the L^2 norm for functions of $x_d\in\mathbb{R}^+$.

If α, β, C do not depend on (τ,η), we have the a priori estimate

$$\alpha\frac{\gamma}{2}\|U_\gamma\|^2+\beta\bigl|U_\gamma(0)\bigr|^2\leqslant C|\varphi_\gamma|^2+\frac{\|S\|_\infty^2}{2\gamma\alpha}\bigl\|\mathcal{F}^{-1}(f)_\gamma\bigr\|^2, \tag{16}$$

where $\gamma=\operatorname{Re}(\tau)$ is fixed, φ, U are the inverse Fourier–Laplace transform of $\widehat{\varphi}$, $\widehat{U}$, generically $g_\gamma=e^{-\gamma t}g$, and the norms are now integrated in both space and time.

Remark. If α depends on (τ, η), we may derive from (15) a priori estimates with loss of derivatives instead of (16).

Typically, $\alpha = \beta = \frac{\gamma^{2s}}{(|\tau|^2+\|\eta\|^2)^s}$ gives

$$\frac{\gamma}{2}\|U_\gamma\|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^d)} + \beta\big|U_\gamma(0)\big|^2_{L^2(\mathbb{R}^+\times\mathbb{R}^{d-1})} \leqslant C|\psi_\gamma|^2_{L^2(\mathbb{R}^+,H^s(\mathbb{R}^{d-1}))} + \frac{\|S\|^2_\infty}{2\gamma}\big\|\mathcal{F}^{-1}(f)_\gamma\big\|^2_{L^2(\mathbb{R}^+;H^s(\mathbb{R}))}.$$

Note that $\mathbb{R}^+ \times \mathbb{R}^d$ is not the usual product of time by space, but $\{x_d > 0\} \times \{x_1, \dots, x_{d-1}, t\}$ (and the same goes for $\mathbb{R}^+ \times \mathbb{R}^{d-1}$).

Proof. Taking the inner product of $S\partial_d\widehat{U} = S(B^{-1}A\widehat{U} + f)$ with $\overline{\widehat{U}}$, integrating on $\mathbb{R}$ and using that S is hermitian gives:

$$\int_0^\infty \widehat{U}^* \cdot SB^{-1}A\widehat{U}\,dx_d + \widehat{U}^*(0)\cdot S\widehat{U}(0) = -\int_0^\infty \widehat{U}^*\cdot Sf \tag{17}$$

for $\widehat{U}$ smooth enough and vanishing at ∞.

Using (13), (14) we obtain:

$$\alpha\gamma\|\widehat{U}\|^2 + \beta\big|\widehat{U}(0)\big|^2 \leqslant C|\widehat{\varphi}|^2 + \|S\|_\infty\|f\|\|\widehat{U}\|, \tag{18}$$

hence (15) by Young's inequality.

Now, if α, β are constants, we integrate this inequality on $\mathbb{R}^d = \{(\delta, \zeta_1, \dots, \zeta_{d-1}) \in \mathbb{R}^d\}$, and since $\mathcal{F}U_\gamma(t, y_1, \dots, y_{d-1}) = \widehat{U}(\gamma + i\delta, \zeta_1, \zeta_{d-1})$, we obtain (16) by Plancherel's theorem. $\square$

In the remainder of this section, we focus on the case $d = 1$. Would there exist a bounded symmetrizer S satisfying an inequality of the kind (13) with α constant, the real part of the eigenvalues of G should remain bounded away from 0 for $\mathrm{Re}(\tau) > 0$ as $|\mathrm{Im}(\tau)| \to \infty$. We are going to show that this is not the case.

Proposition 3. *Let $\tau = \gamma + i\delta$, $\gamma > 0$. Denote by (ω_1^+, ω_2^+), resp. (ω_1^-, ω_2^-) the roots of positive real part, resp. negative, of $P(\cdot\,;\tau)$.*

Two of them have their real part vanishing when $\delta \to +\infty$, while the other two have their real part vanishing when $\delta \to -\infty$.

More precisely, we may choose the numbering of eigenvalues in such a way that:

$$\begin{aligned}
&\mathrm{Re}\big(\omega_1^\pm\big) \sim \pm\frac{\gamma}{2\sqrt{\delta}}, \quad \delta \to +\infty,\\
&\mathrm{Re}\big(\omega_2^\pm\big) \to \pm\infty, \quad \delta \to +\infty,\\
&\mathrm{Re}\big(\omega_1^\pm\big) \to \pm\infty, \quad \delta \to -\infty,\\
&\mathrm{Re}\big(\omega_2^\pm\big) \sim \pm\frac{\gamma}{2\sqrt{|\delta|}}, \quad \delta \to -\infty.
\end{aligned}$$

Proof. Recall the notation $\widehat{\omega}=\frac{\omega}{\sqrt{|\tau|}}$. Using this variable, the equation $P(\omega,\tau)=0$ amounts to (12), which implies $\widehat{\omega}^4+\frac{\tau^2}{|\tau|^2}+o(1)=0$ when $\mathrm{Im}(\tau)=\delta\to\infty$ (by the proof of Lemma 1). Thus:

$$\omega=e\sqrt[4]{-\tau^2}+\mu \quad \text{with } e\in\{1,-1,i,-i\},\ \mu=o\big(\sqrt{|\tau|}\big). \tag{19}$$

Here above $\sqrt[4]{\cdot}$ is defined by the determination of the logarithm on $\mathbb{C}\setminus\mathbb{R}^+$ for which $\mathrm{Log}(-1)=i\pi$.

We study the case $e=1$, i.e. $\omega=\sqrt[4]{-\tau^2}+\mu$ first. When $\delta\to-\infty$, $\mathrm{Re}\sqrt[4]{-\tau^2}\sim\sqrt{|\delta|}\to+\infty$. When $\delta>0$, we have

$$\sqrt[4]{-\tau^2}=\sqrt{\delta}\sqrt[4]{1-\frac{2i\gamma}{\delta}-\frac{\gamma^2}{\delta^2}}.$$

Since $\mathrm{Log}(1-i\varepsilon)=2i\pi-i\varepsilon+o(\varepsilon)$, we have:

$$\mu_0:=\sqrt[4]{-\tau^2}=\sqrt{\delta}i\left(1-\frac{i\gamma}{2\delta}+O\left(\frac{1}{\delta^2}\right)\right)=\sqrt{\delta}\left(i+\frac{\gamma}{2\delta}+O\left(\frac{1}{\delta^2}\right)\right). \tag{20}$$

It suffices to check that $\mathrm{Re}(\mu)=o(\frac{1}{\sqrt{\delta}})$ to complete the proof.

Applying twice the method used to obtain ω_0, we find the asymptotic expansion:

$$\omega_+=\mu_0+\mu_1+\mu_2+o\left(\frac{1}{\sqrt{|\tau|}}\right), \tag{21}$$

with

$$\mu_1=-\frac{\tau\underline{M}}{\mu_0^2},\qquad \mu_2=\frac{-1}{4\mu_0^3}\big(6\mu_0^2\mu_1^2+\big(\underline{M}^2-1\big)\mu_0^2+2\tau\underline{M}\mu_1\big). \tag{22}$$

By (20) and a Taylor expansion again, we obtain that $|\mathrm{Re}(\mu_1)|\lesssim\frac{1}{|\tau|}$ and $\mathrm{Re}(\mu_2)\lesssim\frac{1}{\sqrt{|\tau|^3}}$ (the notation $a\lesssim b$ means that $a\leqslant Cb$ with C a constant independent of the parameters), so that $\mathrm{Re}(\omega_+)=\mathrm{Re}(\mu_0)+o(\frac{1}{\sqrt{|\tau|}})$ as claimed. Thus $\omega=\sqrt[4]{-\tau^2}$ is the ω_1^+ described in Proposition 3.

Now, noticing that the other roots are (essentially) obtained by multiplying ω_1^+ by i, -1 and $-i$, we would get $\omega_1^-=-\omega_1^+$, $\omega_2^+=-i\omega_1^+$, and $\omega_2^-=i\omega_1^+$. $\square$

As the real parts of the eigenvalues of $B^{-1}A$ do not remain bounded away from zero when $\mathrm{Re}(\tau)>0$, the construction of a classical symmetrizer for (8) is compromised. However, we can still search for a generalized symmetrizer as for weakly stable hyperbolic IBVP, see [4] for a general approach. A bigger problem is the very nonhomogeneous structure of $B^{-1}A$ which makes it hard to manipulate. That is why we will use in the next part another equivalent system with better properties.

3. The linearized extended system

In [1], the authors introduce the *extended system*:

$$\begin{cases}\partial_t\zeta+u\cdot\nabla\zeta+a(\zeta)\operatorname{div}u=0,\\ \partial_t u+(u\cdot\nabla)u-\nabla\left(\frac{1}{2}|v|^2\right)-\nabla\big(a(\zeta)\operatorname{div}v\big)=-g'(\zeta)v,\\ \partial_t v+\nabla(u\cdot v)+\nabla\big(a(\zeta)\operatorname{div}u\big)=0,\end{cases} \tag{23}$$

satisfied by ($\zeta = R(\rho)$, u, $v = \nabla\zeta$), if (ρ, u) is a (smooth) solution of (1), with R a primitive of $\rho \to \sqrt{K(\rho)/\rho}$ and $a(\zeta) = \sqrt{\rho K(\rho)}$ (ρ is to be seen here as $R^{-1}(\zeta)$). For simplicity we have written $g'(\zeta)$ for $(g \circ R^{-1})'(\zeta)$.

The main interest of this new system is that it yields a priori estimates without loss of derivatives on the whole space. Indeed, summing the second equation with i times the third, a Schrödinger-type equation appears. As we will see, the analysis of the boundary problem is made possible on the half line too.

Let $(\underline{\zeta}, \underline{u}, \underline{v})$ be a constant state solution of (23), the linearized equations about $(\underline{\zeta}, \underline{u}, \underline{v})$ read:

$$\begin{cases} \partial_t \zeta + \underline{u} \cdot \nabla\zeta + \underline{a} \operatorname{div} u = 0, \\ \partial_t u + (\underline{u} \cdot \nabla) u - \nabla(\underline{v} \cdot v) - \nabla(\underline{a} \operatorname{div} v) = -\underline{g}' v, \\ \partial_t v + \nabla(\underline{u} \cdot v) + \nabla(\underline{a} \operatorname{div} u) = 0, \end{cases} \tag{24}$$

where $a(\underline{\zeta}) = \underline{a}$, $g'(\underline{\zeta}) = \underline{g}'$.

Decomposing $u = (\tilde{u}, u_d)$, $v = (\tilde{v}, v_d)$, with $\tilde{u}$ and $\tilde{v} \in \mathbb{R}^{d-1}$ and making a Fourier–Laplace transform on (23) we get the system:

$$\begin{cases} \underline{u_d}\partial_d\widehat{\zeta} = -\big(\tau\widehat{\zeta} + (\underline{\tilde{u}} \cdot i\eta)\widehat{\zeta} + \underline{a} i\eta \cdot \widehat{\tilde{u}} + \underline{a}\partial_d\widehat{u_d}\big), \\ \underline{u_d}\partial_d\widehat{\tilde{u}} = -\big(\tau\widehat{\tilde{u}} + (\underline{\tilde{u}} \cdot i\eta)\widehat{\tilde{u}} + \underline{g}'\widehat{\tilde{v}} - i\eta\big(\underline{\tilde{v}} \cdot \widehat{\tilde{v}} + \underline{v_d}\widehat{v_d} + \underline{a}(i\eta \cdot \widehat{\tilde{v}}) + \underline{a}\partial_d\widehat{v_d}\big)\big), \\ \partial_d\widehat{u_d} = \partial_d\widehat{u_d}, \\ \underline{a}\partial_d(i\eta \cdot \widehat{\tilde{u}}) + \underline{a}\partial_d^2\widehat{u_d} + \partial_d\underline{\tilde{u}} \cdot \widehat{\tilde{v}} = -\big(\tau\widehat{v_d} + \partial_d(\underline{u_d}\widehat{v_d})\big), \\ 0 = -\big(\tau\widehat{\tilde{v}} + \underline{a} i\eta(i\eta \cdot \widehat{\tilde{u}} + \partial_d\widehat{u_d}) + i\eta(\underline{\tilde{u}} \cdot \widehat{\tilde{v}} + \underline{u_d}\widehat{v_d})\big), \\ \partial_d\widehat{v_d} = \partial_d\widehat{v_d}, \\ \partial_d\underline{\tilde{v}} \cdot \widehat{\tilde{v}} + \underline{a}\partial_d^2\widehat{v_d} = \tau\widehat{u_d} + \underline{\tilde{u}} \cdot i\eta\widehat{u_d} + \underline{u_d}\partial_d\widehat{u_d} + \underline{g}'\widehat{v_d} - \partial_d\underline{v_d}\widehat{v_d}. \end{cases} \tag{25}$$

This is an algebro-differential system in $\widehat{U} = (\widehat{\zeta}, \widehat{u}, \widehat{u_d}, \widehat{v}, \widehat{v_d})$ which can be rewritten as $B\partial_d U = AU$, $U = (\widehat{\zeta}, \widehat{\tilde{u}}, \widehat{u_d}, \partial_d\widehat{u_d}, \widehat{\tilde{v}}, \widehat{v_d}, \partial_d\widehat{v_d})$.

The matrix B is not invertible, except if $d = 1$. In what follows, we restrict ourselves to the one-dimensional case, and we omit the superscript d. The system (25) reduces to

$$\begin{cases} \underline{u}\partial_x\widehat{\zeta} = -(\tau\widehat{\zeta} + \underline{a}\partial_x\widehat{u}), \\ \partial_x\widehat{u} = \partial_x\widehat{u}, \\ \underline{a}\partial_x^2\widehat{u} = -\big(\tau\widehat{v} + \partial_x(\underline{u}\widehat{v})\big), \\ \partial_x\widehat{v} = \partial_x\widehat{v}, \\ \underline{a}\partial_x^2\widehat{v} = \tau\widehat{u} + \underline{u}\partial_x\widehat{u} + \underline{g}'\widehat{v} - \partial_x\underline{v}\widehat{v}, \end{cases}$$

or equivalently $\partial_x U = GU$, with

$$G = \begin{pmatrix} \frac{-\tau}{\underline{u}} & 0 & \frac{-\underline{a}}{\underline{u}} & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & \frac{-\tau}{\underline{a}} & \frac{-\underline{u}}{\underline{a}} \\ 0 & 0 & 0 & 0 & 1 \\ 0 & \frac{\tau}{\underline{a}} & \frac{\underline{u}}{\underline{a}} & \frac{\underline{g}'}{\underline{a}} & \frac{-\underline{v}}{\underline{a}} \end{pmatrix}, \qquad U = \begin{pmatrix} \widehat{\zeta} \\ \widehat{u} \\ \partial_x\widehat{u} \\ \widehat{v} \\ \partial_x\widehat{v} \end{pmatrix}. \tag{26}$$

As in the first part, we may assume $\underline{v} = 0$.

The characteristic polynomial of G is then:

$$\chi_G(X) = \frac{-1}{\underline{a}^2}\left(X + \frac{\tau}{\underline{u_d}}\right)\left((\tau + X\underline{u_d})^2 - \underline{a}X^2\left(\underline{g'} - \underline{a}X^2\right)\right), \tag{27}$$

which is unsurprisingly nearly the same polynomial as the one associated to the original Euler–Korteweg system. Consequently the spectral analysis previously done in Lemma 1 and Proposition 2 still applies.

The form of G is of special interest, because it is very near to the one obtained by Métivier and Zumbrun in [11]: it enjoys a 'quasi-homogeneity' property that we will use to construct symmetrizers. We emphasize this with the following observation.

The restriction of the extended problem to the new variables $V = (\sqrt{|\tau|}\,\widehat{u}, \sqrt{|\tau|}\,\widehat{v}, \partial_x\widehat{u}, \partial_x\widehat{v})$ reads:

$$\partial_x V = \sqrt{|\tau|} H V, \qquad H = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-\tau}{\underline{a}|\tau|} & 0 & \frac{-\underline{u}}{\underline{a}\sqrt{|\tau|}} \\ \frac{\tau}{\underline{a}|\tau|} & \frac{\underline{g'}}{\underline{a}|\tau|} & \frac{\underline{u}}{\underline{a}\sqrt{|\tau|}} & 0 \end{pmatrix}. \tag{28}$$

Remarks.

- It is worth noting that for $|\tau| \gg 1$:

$$H(\tau) = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-\tau}{\underline{a}|\tau|} & 0 & 0 \\ \frac{\tau}{\underline{a}|\tau|} & 0 & 0 & 0 \end{pmatrix} + O(\varepsilon) = H_0\left(\frac{\tau}{|\tau|}\right) + H_1(\varepsilon) \tag{29}$$

  with $\varepsilon = \frac{1}{\sqrt{|\tau|}}$. Whether we focus on medium/low or high frequencies, we will consider H as a matrix depending on τ or on $(\frac{\tau}{|\tau|}, \varepsilon) = (\widehat{\tau}, \varepsilon)$. This method will allow us to consider the area of high frequencies as a compact set (see Fig. 1), and thus obtain uniform estimates. The necessity to study the case $\widehat{\gamma} \to 0$ appears naturally even for $\gamma > 0$ if we let $\delta \to \infty$.
- If there is a forcing term f in (24), the problem after Fourier transform reads $\partial_x V = \sqrt{|\tau|} H V + F$, with $F = (0, f)$. Since it does not complicate the analysis and in sight of further nonlinear analysis, we will study this more general problem.
- Since H is conjugated to $(G_{i,j})_{i \geqslant 2, j \geqslant 2}$, its characteristic polynomial is $\chi_H(X) = (\tau + X\underline{u_d})^2 - \underline{a}X^2(\underline{g'} - \underline{a}X^2)$.
- The first equation in (24) can be seen as an advection equation (with speed $\underline{u}$) for ζ with a forcing term $\underline{a}\operatorname{div} u$. If $\underline{u} > 0$ we have to prescribe ζ at the boundary to solve it, if $\underline{u} < 0$ there is no other boundary condition to add. In what follow we only focus on the existence and regularity of (u, v).

We are now in position to introduce suitable boundary conditions. Let F be a boundary operator as in (8). As we have seen previously, the Lopatinskiĭ condition requires that the number of boundary conditions equal the dimension of H's stable subspace. Applying Lemma 1, we see that this number is two. Thus a natural and simple candidate for the boundary operator F is

$$F = \begin{pmatrix} 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \end{pmatrix}. \tag{30}$$

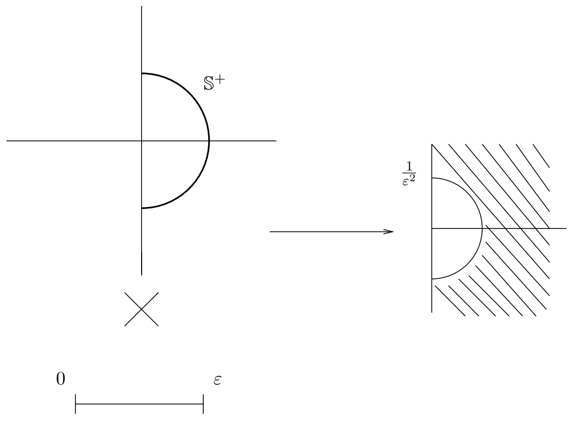

Fig. 1. Correspondence between high frequencies (the hatched area) $\{\mathrm{Re}(\tau) \geqslant 0,\ |\tau| \geqslant \frac{1}{\varepsilon^2}\} \cup \{\infty\}$ and $\mathbb{S}^+ \times [0, \varepsilon]$ via the application $(\widehat{\tau}, s) \to \frac{\widehat{\tau}}{s^2}$.

Denote by Π^- (resp. Π^+) the projector on $E^-(H)$ (resp. $E^+(H)$). The following lemma establishes that F satisfies a refined Lopatinskiĭ condition, in the sense that there is a 'good transversality' of E^- and $\mathrm{Ker}(F)$ as $\mathrm{Im}(\tau) \to \infty$.

Lemma 2. *There exists $\Gamma > 0$ such that for $\mathrm{Re}(\tau) \geqslant \Gamma$ the spaces $\mathrm{Ker}(F)$ and $E^-(\tau)$ are transverse, and*

$$\exists C > 0: \forall X \in \mathbb{R}^4, \quad |X|^2 \leqslant C\left(\left|\Pi^+ X\right|^2 + |FX|^2\right), \tag{31}$$

where C only depends on Γ.

Proof. We use the variables $(\widehat{\tau}, \varepsilon)$ to obtain a uniform inequality. If ω is an eigenvalue of G, we generically denote by V_ω an eigenvector associated to ω. Since we deal with simple eigenvalues when ε is small, this notation will make sense in what follows.

According to the notations of Proposition 3, we extend the space E^- at $\mathrm{Re}(\widehat{\tau}) = 0$ for ε small enough by defining $E^-(\widehat{\tau}, \varepsilon) = \mathrm{vect}(V_{\omega_1^-}, V_{\omega_2^-})$.

The eigenvectors of $H(\widehat{\tau}, 0)$ can be directly computed:

$$V_\omega = \begin{pmatrix} 1 \\ -\frac{a}{\widehat{\tau}}\omega^2 \\ \omega \\ -\frac{a}{\widehat{\tau}}\omega^3 \end{pmatrix}. \tag{32}$$

Obviously, $F : E^- \mapsto \mathrm{Im}(F)$ is an isomorphism if and only if $(FV_{\omega_1^-}, FV_{\omega_2^-})$ is a basis of $\mathbb{C}^2$, or equivalently if $\Delta(\widehat{\tau}) := \det(FV_{\omega_1^-}, FV_{\omega_2^-}) \neq 0$. This last quantity is called a Lopatinskiĭ determinant. We have

$$\Delta(\widehat{\tau}) = \frac{-a}{\widehat{\tau}}\left(\left(\omega_1^-\right)^2 - \left(\omega_2^-\right)^2\right) = \frac{-2a}{\widehat{\tau}}\left(\omega_1^-\right)^2 \neq 0, \tag{33}$$

indeed, for $\varepsilon = 0$, $\omega_2^- = -i\omega_1^-$.

Thus Δ does not vanish, *even for* $\mathrm{Re}(\widehat{\tau}) = 0$.

Now, if $\varepsilon \neq 0$ by continuity of Δ it suffices to choose ε_0 small enough such that the condition $\Delta(\widehat{\tau}, \varepsilon) \neq 0$ remains true for $(\widehat{\tau}, \varepsilon) \in \mathbb{S}^+ \times [0, \varepsilon_0]$, where $\mathbb{S}^+ = \{\tau \in \mathbb{C}\colon\ |\tau| = 1,\ \mathrm{Re}(\tau) \geqslant 0\}$. Since $\mathbb{S}^+ \times [0, \varepsilon_0]$ is a compact set, we obtain

$$\inf_{\mathbb{S}^+ \times [0,\varepsilon_0]} \left|\Delta(\widehat{\tau}, \eta)\right| > 0.$$

This implies that $F : E^- \to \mathbb{R}^2$ is an isomorphism with bounded inverse. Thus we have

$$\begin{aligned} \forall(\widehat{\tau}, \varepsilon) \in \mathbb{S}^+ \times [0, \varepsilon_0],\ \forall X \in E^-(\widehat{\tau}, \varepsilon), \quad |X|^2 &\leqslant 2\left(\left|\Pi^+(X)\right|^2 + \left|\Pi^-(X)\right|^2\right) \\ &\leqslant 2\left(\left|\Pi^+(X)\right|^2 + C\left|F\left(X - \Pi^+(X)\right)\right|^2\right) \\ &\leqslant C'\left(\left|\Pi^+(X)\right|^2 + \left|F(X)\right|^2\right). \end{aligned}$$

This inequality holds for $(\widehat{\tau}, \varepsilon) \in \mathbb{S}^+ \times [0, \varepsilon_0]$, or for the original variable τ when $\mathrm{Re}(\tau) \geqslant 0$ and $|\tau| \geqslant 1/\varepsilon_0^2$. In particular, if we set $\Gamma = 1/\varepsilon_0^2$ the inequality holds for $\tau \in \{\mathrm{Re}(\tau) \geqslant \Gamma\}$. □

Remarks. In fact, we have proven something slightly better than the initial statement: the inequality holds on the set $\{\mathrm{Re}(\tau) \geqslant 0,\ |\tau| \geqslant M\}$ for some M large enough, with an appropriate extension of $E^\pm$ on $\mathrm{Re}(\tau) = 0$.

The lemma is stated for Dirichlet boundary conditions, however it is clear from the proof that the only ingredient is the nonvanishing of the Lopatinskiĭ determinant $\Delta(\widehat{\tau}, 0)$ (which is very easy to compute), this gives a practical criterion to select general boundary conditions. For example, one may check that Neumann boundary conditions also satisfy Lemma 2.

The following proposition is the analogue of Lemmas 2.12, 2.13, 2.14 in [11]:

Proposition 4. *High and medium/low frequencies symmetrizers*:

- *High frequencies: Recall that* $\mathbb{S}^+ = \{\tau \in \mathbb{C}\colon |\tau| = 1,\ \mathrm{Re}(\tau) \geqslant 0\}$. *For all* $\widehat{\tau_0} \in \mathbb{S}^+$, *there are a neighborhood* $\mathcal{V}_\infty(\widehat{\tau_0})$ *of* $(\widehat{\tau_0}, 0)$ *in* $\mathbb{S}^+ \times \mathbb{R}^+$ *and a smooth application* S *on* $\mathcal{V}_\infty(\widehat{\tau_0})$ *with value in the set of self adjoint matrices such that*:

$$\forall(\widehat{\tau}, \varepsilon) \in \mathcal{V}_\infty(\widehat{\tau_0}), \quad \mathrm{Re}\left(SH(\widehat{\tau}, \varepsilon)\right) \geqslant \alpha\left(\mathcal{V}_\infty(\widehat{\tau_0})\right) \mathrm{Re}(\widehat{\tau})\, Id. \tag{34}$$

- *Medium/low frequencies: Fix* M *and* $\Gamma > 0$. *For all* $\tau_0 \in \mathrm{Re}(\tau) \geqslant \Gamma$, $|\tau_0| \leqslant M$, *there are a neighborhood* $\mathcal{V}(\tau_0)$ *of* τ_0 *and a smooth application* S *on* $\mathcal{V}(\tau_0)$ *with value in the set of self adjoint matrices such that*:

$$\forall \tau \in \mathcal{V}(\tau_0), \quad \mathrm{Re}\left(SH(\tau)\right) \geqslant \alpha(\Gamma, M)\, \mathrm{Re}(\tau)\, Id. \tag{35}$$

Moreover, S *can be chosen such that* $S \geqslant \beta I - CF^*F$, *with* c, C' *only depending on* τ_0, Γ *or* $\widehat{\tau_0}, \Gamma$.

Proof. The eigenvalues of $\sqrt{|\tau|}H$ are exactly those of G except $-\tau/\underline{u_d}$. According to Proposition 1 and Lemma 1, they cannot be purely imaginary for $\Gamma > 0$ and there are two eigenvalues of positive real part and two negative ones (counted with multiplicity). Therefore, there is a matrix P on a neighborhood $\mathcal{V}(\tau_0)$ of τ_0 such that

$$P^{-1}HP = \begin{pmatrix} H^+ & 0 \\ 0 & H^- \end{pmatrix}, \tag{36}$$

where $H^\pm$ have their spectrum in $\pm\,\mathrm{Re}(\omega) > 0$.

To deal with low/medium frequencies, it suffices to take S of the form

$$S = \left(P^{-1}\right)^* \begin{pmatrix} kS^+ & 0 \\ 0 & S^- \end{pmatrix} P^{-1} \tag{37}$$

with $S^\pm = \int_{-\infty}^0 e^{\pm tH^{\pm}*} e^{\pm tH^\pm}\, dt$ given by the Lyapunov matrix theorem, and $k > 0$. Indeed,

$$SH = P^{-1*} \begin{pmatrix} kI & 0 \\ 0 & I \end{pmatrix} P^{-1} > 0,$$

and since $\{\mathrm{Re}(\tau) \geqslant \Gamma,\ |\tau| \leqslant M\}$ is compact the eigenvalues of H remain bounded away from zero, thus S is bounded.

We note that by construction,

$$\forall X \in \mathbb{R}^4, \quad \langle SX, X\rangle = \left\langle DP^{-1}X, P^{-1}X\right\rangle$$

with $D = \begin{pmatrix} kS^+ & \\ & S^- \end{pmatrix}$ and $S^\pm$ are positive/negative symmetric matrices.

We thus have

$$\langle SX, X\rangle \geqslant ck\left|\Pi^+ X\right|^2 - C_1\left|\Pi^- X\right|^2,$$

and, by Lemma 2,

$$\left|\Pi^+ X\right|^2 \geqslant \frac{|X|^2}{C} - |FX|^2 \quad \Rightarrow \quad \langle SX, X\rangle \geqslant \beta|X|^2 - C|FX|^2, \tag{38}$$

provided k is large enough (i.e.: $\frac{ck}{C} > C_1|\Pi_-|_{L^\infty(\mathcal{V}(\tau_0))}$), so S satisfies the second inequality too.

To deal with high frequencies, we distinguish the cases $\mathrm{Re}(\widehat{\tau_0}) > 0$ and $\mathrm{Re}(\widehat{\tau_0}) = 0$.

• If $\mathrm{Re}(\widehat{\tau_0}) > 0$, the eigenvalues of $H(\widehat{\tau_0}, 0)$ are not purely imaginary and are distinct. By continuity of the roots, there are a neighborhood of $(\widehat{\tau_0}, 0)$ in $\mathbb{S}^+ \times \mathbb{R}^+$ and applications $P, H^\pm(\widehat{\tau}, \varepsilon)$ allowing to define S as previously.

• If $\mathrm{Re}(\widehat{\tau_0}) = 0, \varepsilon = 0$ (i.e.: $\widehat{\tau_0} = \pm i$), two of the roots are purely imaginary, so that we cannot sort the roots as previously.

However, by Proposition 3, $\sqrt{|\tau|}H$ has four *distinct* eigenvalues $(\omega_1^\pm, \omega_2^\pm)$, with $\mathrm{Re}(\omega_2^\pm) = 0$ if $\mathrm{Re}(\widehat{\tau}) = 0$. Thus, H is smoothly diagonalizable on a neighborhood of $(\pm i, 0)$:

$$\exists P(\tau):\quad P^{-1}HP = \frac{1}{\sqrt{|\tau|}} \begin{pmatrix} \omega_1^+ & & & \\ & \omega_2^+ & & \\ & & \omega_1^- & \\ & & & \omega_2^- \end{pmatrix}, \tag{39}$$

with

$$\mathrm{Re}\left(\omega_2^\pm\right) \asymp \frac{\pm\gamma}{\sqrt{|\delta|}} \asymp \pm\widehat{\gamma}\sqrt{|\tau|}, \tag{40}$$

the notation $x \asymp y$ meaning that there exist $\beta \geqslant \alpha > 0$ such that $\alpha x \leqslant y \leqslant \beta x$.

If we set

$$S = P^{-1*}\begin{pmatrix} k & & & \\ & k & & \\ & & -1 & \\ & & & -1 \end{pmatrix} P^{-1},$$

obviously $SH \geqslant \alpha\widehat{\gamma}$. Moreover, similarly to the proof of Lemma 2, we can extend the projectors $\Pi^{\pm}$ to (i,ε) for ε small enough. Thus $\langle SX, X\rangle \geqslant ck|\Pi^+X|^2 - C_1|\Pi^-X|^2$ makes sense on a neighborhood $\mathcal{V}_\infty(i,0)$. But according to the remark after Lemma 2, on $\mathcal{V}_\infty(i,0)$:

$$|X|^2 \leqslant C\big(\big|\Pi^+X\big| + |FX|^2\big) \quad\Rightarrow\quad \langle SX, X\rangle \geqslant \frac{ck}{C}|X|^2 - C_1\big|\Pi^-X\big|^2 - ck|FX|^2$$

and again, it suffices to choose k large enough to conclude. □

Remarks. The matrices exhibited are not like the ones obtained in the usual theory of homogeneous hyperbolic IBVP. Indeed, for high frequencies we just have $\mathrm{Re}(SH) \geqslant \alpha\,\mathrm{Re}(\widehat{\tau})I_d$ (and so, $\mathrm{Re}(S\sqrt{|\tau|}H) \geqslant \alpha\frac{\mathrm{Re}(\tau)}{\sqrt{|\tau|}}$ instead of $\mathrm{Re}(S\sqrt{|\tau|}G_1) \geqslant \alpha\,\mathrm{Re}(\tau)$).

The results of Proposition 4 and Lemma 2 can be summarized with the following assertion:

Let γ be a real positive number. There exists a self adjoint bounded operator $S(\tau)$ for $\tau \in \gamma + i\mathbb{R}$ for the problem (8) satisfying the estimates

$$\sqrt{|\tau|}SH \geqslant \alpha\frac{\gamma}{\sqrt{|\tau|}}, \tag{41}$$

$$S \geqslant \beta I - CF^*F. \tag{42}$$

Indeed, for any $\tilde{\tau} \in \mathbb{S}^+$, there is a neighborhood $\mathcal{V}_\infty(\tilde{\tau}) \subset \mathbb{S}^+ \times \mathbb{R}^+$ as in Proposition 4 (high frequencies). By compactness of $\mathbb{S}^+$, we obtain a finite family $(\mathcal{V}_\infty(\widehat{\tau_i}))_{1\leqslant i\leqslant n}$ such that the projections of $\mathcal{V}_\infty(\widehat{\tau_i})$ on $\mathbb{S}^+$ cover it.

Up to taking smaller $\mathcal{V}_\infty(\widehat{\tau_i})$, we may assume that they have the form $\{e^{i\theta},\ \theta_i \leqslant \theta \leqslant \theta_i'\} \times [0,\varepsilon_i]$. Let $R \in \mathbb{R}^{+*}$ be greater than $\frac{1}{\min(\varepsilon_i^2)}$. We can obtain a finite covering of $\{\tau\colon\ \mathrm{Re}(\tau) \geqslant \Gamma \text{ and } |\tau| \leqslant R\}$ by sets $(\mathcal{V}(\tau_j))_{1\leqslant j\leqslant m}$ as in Proposition 4 (low/medium frequencies). Since $(\mathcal{V}_\infty(\widehat{\tau_i})) \cup (\mathcal{V}(\tau_j))$ is a finite covering of $\{\mathrm{Re}(\tau) \geqslant \Gamma\}$ (see Fig. 2), using a partition of the unity, we finally obtain an operator S on the whole set $\{\tau\colon\ \mathrm{Re}(\tau) \geqslant \Gamma\}$ satisfying (41).

Finally, Proposition 2 gives:

Theorem 1. *Let Γ be a real positive number, $\exists C > 0$ such that for all $\tau \in \mathrm{Re}(\tau) \geqslant \Gamma$, and V the solution of*

$$\begin{cases} \partial_x V = \sqrt{|\tau|}HV + f, \\ FV(0) = \psi, \end{cases} \tag{43}$$

we have

$$\frac{\alpha\gamma}{2\sqrt{|\tau|}}\|V\|^2 + \beta\big|V(0)\big|^2 \leqslant C|\varphi|^2 + \frac{\sqrt{|\tau|}}{2\alpha\gamma}\|f\|^2. \tag{44}$$

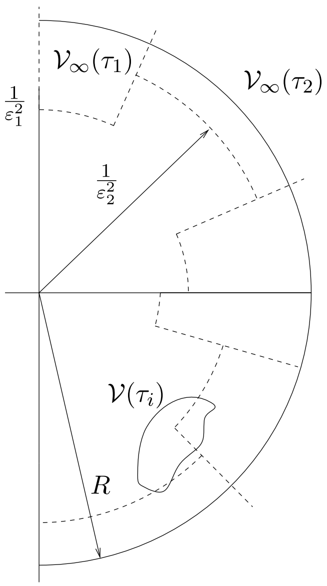

Fig. 2. High frequencies are in the (truncated) cones, low/medium inside the half circle.

In term of the original unknowns, Theorem 1 reads

$$\begin{aligned}&\frac{\gamma\sqrt{|\tau|}}{2}\left\|(\widehat{u},\widehat{v})\right\|^2+\frac{\gamma}{2\sqrt{|\tau|}}\left\|\partial_x(\widehat{u},\widehat{v})\right\|^2+\beta|\tau|\left|(\widehat{u},\widehat{v})(0)\right|^2+\beta\left|\partial_x(\widehat{u},\widehat{v})(0)\right|^2\\&\quad\leqslant C|\tau||\widehat{\varphi}|^2+\frac{|\sqrt{\tau}|}{2\gamma}\|0,f\|^2.\end{aligned}\tag{45}$$

In order to derive higher order estimates, we introduce the weighted Sobolev spaces

$$H_\gamma^s(\mathbb{R})=\left\{f\colon\ \|f\|_{s,\gamma}=\left(\int\left|\widehat{e^{-\gamma\cdot}f}(\delta)\right|^2\left(\gamma^2+\delta^2\right)^s d\delta\right)^{\frac{1}{2}}<\infty\right\}.\tag{46}$$

These norms appear naturally because of the coefficient $\sqrt{|\tau|}$ in our inequalities. Note in particular that $\sqrt{|\tau|}=(\gamma^2+\delta^2)^{\frac{1}{4}}$.

By integration in $\mathrm{Im}(\tau)$ and use of the Plancherel theorem, the previous estimate gives an inequality in term of these norms:

$$\begin{aligned}&\frac{\gamma}{2}\left\|(u,v)\right\|^2_{L_x^2(H^{1/2}_{\gamma,t})}+\frac{\gamma}{2}\left\|\partial_x(u,v)\right\|_{L^2_{x,t}}+\beta\left\|(u,v)(0)\right\|_{H^{3/4}_{\gamma,t}}+\beta\left\|\partial_x(u,v)(0)\right\|_{H^{1/4}_{\gamma,t}}\\&\quad\leqslant C\|\varphi\|^2_{H^{3/4}_{\gamma,t}}+\frac{1}{2\gamma}\|f\|_{L_x^2(H^{1/2}_{\gamma,t})}.\end{aligned}\tag{47}$$

4. Existence and uniqueness results

In this section we shall use the previous construction of symmetrizers to actually prove the well-posedness of IBVP for the extended system. Actually, it allows us to solve the problem with zero initial data, and the general IBVP is treated by proving trace results for an explicit solution of the Cauchy problem on the whole line.

Theorem 2. *For any $T > 0$, there exists $C(T)$ such that the problem*

$$\begin{cases} \partial_t u + \underline{u}\partial_x u - \underline{a}\partial_x^2 v = -\underline{g}' v, & \\ \partial_t v + \underline{u}\partial_x v + \underline{a}\partial_x^2 u = 0, & (x,t) \in \mathbb{R}^+ \times [0,T], \\ (u,v)|_{t=0} = (u_0, v_0) \in H^1(\mathbb{R}^+), & x \in \mathbb{R}^+, \\ (u,v)|(0) = \varphi \in H^{3/4}([0,T]), & t \in [0,T], \end{cases} \tag{48}$$

admits a unique solution in $L^2([0,T]; H^1(\mathbb{R}^+)) \cap L^2(\mathbb{R}^+; H^{1/2}([0,T]))$ if $\varphi(0) = (u_0(0), v_0(0))$.

Moreover, this solution satisfies the estimate:

$$\begin{aligned} &\|(u,v)\|_{L^2(\mathbb{R}^+;H^{1/2}([0,T]))} + \|\partial_x(u,v)(0)\|_{L^2(\mathbb{R}^+\times[0,T])} + \|(u,v)(0)\|_{H^{3/4}([0,T])} \\ &\quad + \|\partial_x(u,v)(0)\|_{H^{1/4}([0,T])} \\ &\leqslant C(T)\big(\|(u_0,v_0)\|_{H^1(\mathbb{R}^+)} + \|\varphi\|_{H^{3/4}([0,T])}\big). \end{aligned} \tag{49}$$

Proof. *Existence:* The proof is in two parts: we first give the solution of a Cauchy problem, and check that its restriction to $[0,T] \times \mathbb{R}^+$ satisfies the appropriate estimate. To do it, we will adapt a trace regularity result (see Fokas and Sung [5], Lemma 5.1) for scalar PDEs on the real line to our case. In the second part, we find a solution of the pure boundary value problem. By linearity this gives a solution to the IBVP.

Let $(u_0, v_0) \in H^1(\mathbb{R}^+)$ be the initial data. We extend them continuously as functions in $H^1(\mathbb{R})$ still written (u_0, v_0), and then we solve the Cauchy problem:

$$\begin{cases} \partial_t u + \underline{u}\partial_x u - \underline{a}\partial_x^2 v + \underline{g}' v = 0, & \\ \partial_t v + \underline{u}\partial_x v + \underline{a}\partial_x^2 u = 0, & t \in [0,T],\ x \in \mathbb{R}, \\ (u,v)|_{t=0} = (u_0, v_0), & x \in \mathbb{R}. \end{cases}$$

It is equivalent after a Fourier transform in space to solve $\partial_t U - AU = 0$, $U|_{t=0} = U_0 = (\widehat{u_0}, \widehat{v_0})$, where $U = (\widehat{u}, \widehat{v})$, $A = \begin{pmatrix} -i\underline{u}\zeta & -\underline{a}\zeta^2 - \underline{g}' \\ \underline{a}\zeta^2 & -i\underline{u}\zeta \end{pmatrix}$.

Therefore, $U = e^{At}U_0$, and $(u,v) = \int_{\mathbb{R}} e^{ix\zeta} e^{At} U_0 \, d\zeta$. The matrix $A(\zeta)$ has two distinct purely imaginary eigenvalues, except for $\zeta = 0$:

$$\lambda_\pm = -i\underline{u}\zeta \pm i\sqrt{\underline{a}\zeta^2\big(\underline{a}\zeta^2 + \underline{g}'\big)}.$$

Let

$$P = \begin{pmatrix} 1 & 1 \\ \frac{\lambda_+ + i\underline{u}\zeta}{\underline{a}\zeta^2 + \underline{g}'} & \frac{\lambda_- + i\underline{u}\zeta}{\underline{a}\zeta^2 + \underline{g}'} \end{pmatrix}.$$

Given $R > 0$, one can check that $P(\zeta)$ is invertible on $]-\infty, -R] \cup [R, \infty[$, such that $P^{-1}AP = \operatorname{diag}(\lambda_+, \lambda_-)$ and P, P^{-1} are bounded. Thus $(t,\zeta) \to e^{A(\zeta)t}$ remains bounded on $[0,\infty[\times (]-\infty, -R] \cup [R, \infty[)$.

The first consequence is that $(u, v) \in L^2([0,T]; H^1(\mathbb{R}^+))$:

$$\begin{aligned}\|(u,v)\|^2_{L^2([0,T];H^1(\mathbb{R}^+))} &\leqslant \int_0^T \int_{\mathbb{R}} \big(\big|e^{A(\zeta)t}\big|\big|U_0(\zeta)\big|\big)^2\big(1+\zeta^2\big)\,d\zeta\,dt \\ &\leqslant C\int_0^T \|u_0, v_0\|^2_{H^1(\mathbb{R}^+)} \\ &\leqslant CT\|u_0, v_0\|^2_{H^1(\mathbb{R}^+)}\end{aligned}$$

and we will now focus on the regularity of the trace at $x=0$.

We first note that for R large enough, $\zeta \in \mathbb{R}^* \to \lambda_\pm(\zeta)/i$ is a diffeomorphism on $]-\infty, R]$ and $]R, +\infty[$, with the asymptotic behavior $|\lambda_\pm(\zeta)| \sim \underline{a}\zeta^2$, $|\lambda'_\pm(\zeta)| \sim 2\underline{a}|\zeta|$. Therefore, the solution will be split into three terms:

$$(u,v)(x,t) = \int_{-\infty}^{-R} e^{ix\zeta} e^{At} U_0\,d\zeta + \int_{-R}^{R} e^{ix\zeta} e^{At} U_0\,d\zeta + \int_{R}^{\infty} e^{ix\zeta} e^{At} U_0\,d\zeta. \tag{50}$$

The second term obviously defines a C^∞ function, thus we just need to study the regularity of the two others. In fact, we will only study the third one, the analysis of the first being exactly the same.

Since $P\big(\begin{smallmatrix}\lambda_+ & 0\\ 0 & \lambda_-\end{smallmatrix}\big)P^{-1} = A$, we have

$$\int_R^\infty e^{ix\zeta} e^{At} U_0\,d\zeta = \int_R^\infty e^{\lambda_+ t} B_+ U_0 + e^{\lambda_- t} B_- U_0\,d\zeta,$$

where $B_\pm(x,\zeta)$ are bounded matrices. Since $\lambda_\pm/i$ are diffeomorphism on $[R,\infty[$, we can use the changes of variable $\zeta = (\lambda_\pm/i)^{-1}(\eta_\pm)$ to get

$$\begin{aligned}\int_R^\infty e^{\lambda_+ t} B_+ U_0 + e^{\lambda_- t} B_- U_0\,d\zeta = {} & \int_{(\lambda_+/i)(R)}^{\infty} e^{i\eta_+ t} B_+\big((\lambda_+/i)^{-1}(\eta_+)\big)\frac{U_0\,d\eta_+}{(\lambda_+/i)' \circ (\lambda_+/i)^{-1}(\eta_+)} \\ & + \int_{(\lambda_-/i)(R)}^{-\infty} e^{i\eta_- t} B_-\big((\lambda_-/i)^{-1}(\eta_-)\big)\frac{U_0\,d\eta_-}{(\lambda_-/i)' \circ (\lambda_-/i)^{-1}(\eta_-)}.\end{aligned}$$

These integrals can be seen as (truncated) inverse Fourier transforms in t, so the trace of (u, v) at $x=0$ is in $H^{3/4}$ if

$$\int_{(\lambda_+/i)(R)}^{\infty} \big(1+|\eta_+|\big)^{3/2}\big|e^{i\eta_+ t} B_+\big((0, \lambda_+/i)^{-1}(\eta_+)\big)U_0\big|^2 \frac{d\eta_+}{|(\lambda_+/i)' \circ (\lambda_+/i)^{-1}(\eta_+)|^2} < \infty,$$

$$\int_{(\lambda_-/i)(R)}^{\infty} \big(1+|\eta_-|\big)^{3/2}\big|e^{i\eta_+ t} B_-\big(0, (\lambda_-/i)^{-1}(\eta_-)\big)U_0\big|^2 \frac{d\eta_-}{|(\lambda_-/i)' \circ (\lambda_-/i)^{-1}(\eta_-)|^2} < \infty.$$

Going back to the initial variables and using the asymptotic behavior of $\lambda_\pm, \lambda'_\pm$, we get

$$\begin{aligned}
&\int\limits_{(\lambda_\pm/i)(R)}^{\pm\infty} \big(1+|\eta_\pm|\big)^{3/2}\big|B_\pm\big((0,\lambda_\pm/i)^{-1}(\eta_\pm)\big)U_0\big|^2 \frac{d\eta_\pm}{|(\lambda_\pm/i)'\circ(\lambda_\pm/i)^{-1}(\eta_\pm)|^2} \\
&= \int\limits_R^\infty \big(1+\big|\lambda_\pm(\zeta)\big|\big)^{3/2}\big|B_\pm(0,\zeta)U_0\big|^2\frac{d\zeta}{|\lambda'_\pm(\zeta)|} \\
&\leqslant C\int\limits_R^\infty \big(1+|\zeta|\big)^3|U_0|^2\frac{d\zeta}{1+|\zeta|} \\
&\leqslant C\big\|(u_0,v_0)\big\|^2_{H^1}.
\end{aligned}$$

Finally, we have the estimate

$$\big\|(u,v)|(0)\big\|_{H^{3/4}([0,T])} \leqslant C(T)\big\|(u_0,v_0)\big\|_{H^1(\mathbb{R}^+)} \tag{51}$$

and the same kind of argument leads to

$$\big\|\partial_x(u,v)|_{x=0}\big\|_{H^{1/4}([0,T])} \leqslant C(T)\big\|(u_0,v_0)\big\|_{H^1(\mathbb{R}^+)} \tag{52}$$

(since $e^{A(\zeta)T}=O(T)$ on the neighborhood of $\zeta=0$, the contribution of the second term may not be bounded as $T\to\infty$).

We now prove that $(u,v)\in L^2_x(\mathbb{R}:H^{1/2}([0,T]))$. First write

$$(u,v)=\int\limits_{-\infty}^{\infty} e^{ix\zeta}\big(e^{\lambda_+ t}B_+ + e^{\lambda_- t}B_-\big)U_0(\zeta)\,d\zeta. \tag{53}$$

We have by definition, and using the Plancherel equality:

$$\begin{aligned}
\big\|(u,v)\big\|_{L^2_x(H^{1/2}([0,T]))} &= \big\|(u,v)\big\|^2_{L^2_x(L^2_t)} + \int\limits_{\mathbb{R}}\iint\limits_{[0,T]^2} \frac{|(u,v)(x,t)-(u,v)(x,s)|^2}{|t-s|^2}\,dt\,ds\,dx \\
&= \big\|(u,v)\big\|^2_{L^2_x(L^2_t)} \\
&\quad + \iint\limits_{[0,T]^2}\int\limits_{\mathbb{R}} \frac{|(e^{\lambda_+ t}B_+ + e^{\lambda_- t}B_- - e^{\lambda_+ s}B_+ - e^{\lambda_- s}B_-)U_0(\zeta)|^2}{|t-s|^2}\,d\zeta\,dt\,ds.
\end{aligned}$$

If $\lambda_\pm\neq 0$:

$$\iint\limits_{[0,T]^2} \frac{|(e^{\lambda_\pm t}-e^{\lambda_\pm s})|^2}{|t-s|^2}\,ds\,dt = \iint\limits_{[0,\lambda_\pm T/i]^2} \frac{|(e^{it'}-e^{is'})|^2}{|t'-s'|^2}\,ds'\,dt'$$

$$\leqslant \int_0^{\lambda_\pm T/i} \int_{-\infty}^{\infty} \frac{|(e^{it'}-1)|^2}{|t'|^2}\,dt'\,ds'$$

$$\leqslant |\lambda_\pm T| \int_{-\infty}^{\infty} \frac{|(e^{it'}-1)|^2}{|t'|^2}\,dt',$$

and the inequality is obviously true for the special case $\lambda_\pm = 0$. Using this, the boundedness of $B_\pm$ and the estimate $|\lambda_\pm(\zeta)| = O(\zeta^2)$ when $|\zeta| \to \infty$, we obtain

$$\|(u,v)\|_{L^2_x(H^{1/2}([0,T]))} \leqslant C \int_{\mathbb{R}} |U_0|^2\left(1+\zeta^2\right) d\zeta \leqslant C' \|(u_0,v_0)\|^2_{H^1(\mathbb{R}^+)}. \tag{54}$$

To sum it up, the restriction to $\mathbb{R}^+$ of the solution of the Cauchy problem satisfies

$$\begin{aligned} &\|(u,v)\|_{L^2(\mathbb{R}^+;H^{1/2}([0,T]))} + \|\partial_x(u,v)\|_{L^2(\mathbb{R}^+\times[0,T])} + \|(u,v)\|_{H^{3/4}([0,T])} + \|\partial_x(u,v)\|_{H^{1/4}([0,T])} \\ &\quad \lesssim \|(u_0,v_0)\|_{H^1(\mathbb{R}^+)}. \end{aligned}$$

Now, since we have a solution of the Cauchy problem with control of the boundary data, by linearity we are with a pure boundary value problem (BVP):

$$\begin{cases} \partial_t u + \underline{u}\partial_x u - \underline{a}\partial_x^2 v + \underline{g'} v = 0, \\ \partial_t v + \underline{u}\partial_x v + \underline{a}\partial_x^2 u = 0, & t \in \mathbb{R},\ x \geqslant 0, \\ (u,v)(0) = \varphi, & t \in \mathbb{R}, \end{cases} \tag{55}$$

where φ has been continuously extended on $\mathbb{R}$ as an $H^{3/4}$ function with $\varphi = 0$, $t < 0$.

To solve it, we now use a Fourier transform in time which gives the ODE:

$$\partial_x V = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & \frac{-\tau}{\underline{a}} & 0 & \frac{-\underline{u}}{\underline{a}} \\ \frac{\tau}{\underline{a}} & \frac{\underline{g'}}{\underline{a}} & \frac{\underline{u}}{\underline{a}} & 0 \end{pmatrix} V, \qquad FV(0) = \widehat{\varphi}, \qquad V(0) \in E^-(\tau). \tag{56}$$

Here, $\tau = \gamma + i\delta$ for some $\gamma > 0$, and generically $\widehat{f} = \int_{\mathbb{R}} e^{-\tau t} f(t)\,dt$.

Since F is an isomorphism $E^- \to \mathbb{R}^2$, for any fixed τ, the ODE problem (56) has a unique solution. To begin with, we may assume that φ has compact support, so that $\widehat{g}$ is τ-holomorphic and V as well.

We define the weighted L^2 spaces by $\|h\|^2_{L^{2,s}} = \int |h|^2 |\tau|^{2s}\,d\tau$, and we will assume that $g \in L^{2,3/4}$. We use $\|j\|^2_{L^2_x(L^{2,s})} = \int |j|^2|\tau|^{2s}\,dx\,d\tau$ too.

By construction, for all τ the solution $V(\cdot,\tau)$ belongs to $H^1(\mathbb{R}^+)$, thus it admits a trace $V(0,\tau)$ holomorphic in τ; we can apply the estimate (44), and integrate in τ to obtain the "Fourier version" of (47). If $V = (V_1, V_2) \in \mathbb{R}^2 \times \mathbb{R}^2$,

$$\alpha\frac{\gamma}{2}\|V_1\|^2_{L^2_x(L^{2,1/2})} + \alpha\frac{\gamma}{2}\|V_2\|_{L^2_x(L^2)} + \beta\|V_1(0)\|_{L^{2,3/4}} + \beta\|V_2(0)\|_{L^{2,1/4}} \leqslant C\|\widehat{\varphi}\|^2_{L^{2,3/4}}. \tag{57}$$

In particular, $\|V\|_{L_x^2(L^2)}$ is bounded; the Paley–Wiener theorem gives the existence of a function $U \in L^2_{x,t}$ such that $\widehat{U} = V$, $U \equiv 0$ if $t < 0$. By inverse Fourier transform, $U_1 = \mathcal{F}^{-1}(V_1)$ is a solution of (55), and as $U_1 \equiv 0$, $t < 0$, the trace of U_1 is null at $t = 0$.

Finally, U is a solution of the boundary value problem since by inverse Fourier transform $V_1(0,\tau) = \widehat{\varphi} \Rightarrow U(0,t) = \varphi$.

Now, we allow φ not to have a compact support: by truncature/regularization, we find φ_n with compact support such that $\widehat{\varphi_n} \to \widehat{\varphi}$ in $L^{2,3/4}$. Let V^n be the solution of (56) with data $\widehat{\varphi_n}$, and U^n the associated solution of (24). Using the Fourier version of (47), we see that V^n has a limit $V = (V_1, V_2) \in L_x^2(L^{2,1/2}) \times L_x^2(L^{2,0})$, and $V^n(0)$ has a limit $(W^1(\tau), W^2(\tau)) \in L^{2,3/4} \times L^{2,1/2}$ (and similar convergence for U^n). We prove that (W^1, W^2) is the trace at $x = 0$ of (V_1, V_2).

By convergence of V^n, V is an L_x^2 solution of $\partial_x V = GV$ in $\mathcal{D}'(\mathbb{R} \times \mathbb{R}^{+*})$, but GV also belongs to L_x^2, thus V admits a trace at $x = 0$ defined by $V(0) = \int_0^\infty V \partial_x V\,dx$, which is in $L^{2,-1/2}$. By convergence of $V^n(0)$ and uniqueness of the limit, we conclude that $(V_1(0), V_2(0)) \in L^{2,1/2} \times L^2$, $V_1(0) = \widehat{g}$, and $U|_{[0,T]}$ is a solution of the pure boundary value problem satisfying:

$$\begin{aligned}
&\big\|(u,v)\big\|_{L^2([0,T];H^1(\mathbb{R}^+))} + \big\|(u,v)(0)\big\|_{H^{3/4}([0,T])} + \big\|\partial_x(u,v)(0)\big\|_{H^{1/4}([0,T])} \\
&\quad \leqslant C\Big(\big\|(u,v)\big\|^2_{L_x^2(H^{1/2}_{\gamma,t})} + \big\|\partial_x(u,v)\big\|_{L^2_{x,t}} + \big\|(u,v)(0)\big\|_{H^{3/4}_{\gamma,t}} + \big\|\partial_x(u,v)(0)\big\|_{H^{1/4}_{\gamma,t}}\Big) \\
&\quad \leqslant C'\|\varphi\|_{H^{3/4}_\gamma(\mathbb{R})} \\
&\quad \leqslant C''\|\varphi\|_{H^{3/4}([0,T])}.
\end{aligned}$$

Uniqueness: The uniqueness is a consequence of the existence of a priori estimates, see for example [3] for a proof in the case of homogeneous first order systems, which can be directly adapted to our case. □

If the data are more regular, higher regularity can be obtained in anisotropic spaces. We define

$$H^{m,2}\big(\mathbb{R}^+ \times [0,T]\big) = \Big\{u:\ \forall 0 \leqslant k \leqslant m,\ \partial_x^k u \in L^2\Big(\mathbb{R}^+; H^{\frac{m-k}{2}}\big([0,T]\big)\Big)\Big\},$$

and the analogous weighted spaces

$$H^{m,2}_\gamma\big(\mathbb{R}^+ \times \mathbb{R}\big) = \Big\{u:\ \forall 0 \leqslant k \leqslant m,\ \partial_x^k u \in L^2\Big(\mathbb{R}^+; H_\gamma^{\frac{m-k}{2}}(\mathbb{R})\Big)\Big\}.$$

Since we deal with higher levels of regularity, we require that the data satisfy the following compatibility condition: if (u_c, v_c) is the solution of the Cauchy problem on $[0,T]$ (as in the previous proof), the function $\varphi - (u_c, v_c)|_{x=0}$ is in $H^{m/2+1/4}([0,T])$ and the extension defined by

$$\begin{cases}
\varphi - (u_c, v_c)|_{x=0} = 0 & \text{for } t < 0,\\
\varphi - (u_c, v_c)|_{x=0} = \psi & \text{with } \psi \text{ a compactly supported smooth extension,}
\end{cases}$$

must be in $H_\gamma^{m/2+1/4}(\mathbb{R})$ (a simpler – but less accurate – condition would be to assume that φ, resp. (u_0, v_0), are flat at $t = 0$, resp. $x = 0$).

Corollary 1. *Let $m \in \mathbb{N}$, $m \geqslant 2$, and (u,v) be the unique solution of*

$$\begin{cases}
\partial_t u + \underline{u}\partial_x u - \underline{a}\partial_x^2 v = -\underline{g}' v, & \\
\partial_t v + \underline{u}\partial_x v + \underline{a}\partial_x^2 u = 0, & (x,t) \in \mathbb{R}^+ \times [0,T],\\
(u,v)|_{t=0} = (u_0, v_0), & x \in \mathbb{R}^+,\\
(u,v)|(0) = \varphi, & t \in [0,T].
\end{cases}$$

If $(u_0, v_0) \in H^m(\mathbb{R}^+)$ *and* $\varphi \in H^{m/2+1/4}$ *then*

$$u \in H^{m,2}\big(\mathbb{R}^+ \times [0,T]\big) \quad \text{and} \quad (\partial_x u, \partial_x v)(0) \in H^{(m-1)/2+1/4}\big([0,T]\big).$$

Proof. It is sufficient to check that the solution constructed in the previous proof has the desired regularity. We shortly describe how this is done for the Cauchy problem and the pure boundary value problem. We keep the notations of the previous proof.

• Cauchy problem: we first study space–time regularity. From $U = e^{A(\zeta)t}U_0(\zeta)$ we get

$$(u,v) \in L^2\big([0,T]; H^m\big(\mathbb{R}^+\big)\big).$$

The equations

$$\begin{cases} \partial_t u + \underline{u}\partial_x u - \underline{a}\partial_x^2 v = -\underline{g}' v, \\ \partial_t v + \underline{u}\partial_x v + \underline{a}\partial_x^2 u = 0, \qquad (x,t) \in \mathbb{R}^+ \times [0,T], \end{cases}$$

imply (by spatial regularity of (u,v)) that $(\partial_t u, \partial_t v) \in L^2([0,T] \times \mathbb{R}^+)$. Differentiating in t and x and applying again the argument we obtain $\forall (k,l) \in \mathbb{N}$ such that $k + 2l \leqslant m$, $\partial_t^l \partial_x^k u \in L^2([0,T] \times \mathbb{R}^+)$.

Now if $(m-k)/2 \notin \mathbb{N}$, we write $\frac{m-k}{2} = l + \frac{1}{2}$, to complete the regularity we have to check that $\partial_t^l \partial_x^k (u,v) \in L^2(\mathbb{R}^+; H^{1/2}([0,T]))$, or equivalently (as in the proof of the previous theorem):

$$\iint\limits_{[0,T]^2} \int\limits_{\mathbb{R}} \frac{|(e^{\lambda_+ t}B_+ + e^{\lambda_- t}B_- - e^{\lambda_+ s}B_+ - e^{\lambda_- s}B_-)U_0(\zeta)|^2}{|t-s|^2} |\zeta|^{2k} \big|A(\zeta)\big|^{2l} \, d\zeta \, dt \, ds < \infty.$$

Recall that $A(\zeta) = O(\zeta^2)$, using $k + 2l + 1 = m$ we get:

$$\begin{aligned} &\iint\limits_{[0,T]^2} \int\limits_{\mathbb{R}} \frac{|(e^{\lambda_+ t}B_+ + e^{\lambda_- t}B_- - e^{\lambda_+ s}B_+ - e^{\lambda_- s}B_-)U_0(\zeta)|^2}{|t-s|^2} |\zeta|^{2k} \big|A(\zeta)\big|^{2l} \, d\zeta \, dt \, ds \\ &\quad \leqslant C \int\limits_{\mathbb{R}} \big(1 + |\zeta|^2\big)^{k+2l} \big(1 + |\zeta|^2\big) \big|U_0(\zeta)\big|^2 \, d\zeta \\ &\quad \leqslant C' \|U_0\|_{H^m}. \end{aligned}$$

For the boundary regularity, the crucial estimate was

$$\begin{aligned} \int\limits_R^\infty \big(1 + \big|\lambda_\pm(\zeta)\big|\big)^{3/2} \big|B_\pm(0,\zeta)U_0\big|^2 \frac{d\zeta}{|\lambda'_\pm(\zeta)|} &\leqslant C \int\limits_R^\infty \big(1 + |\zeta|\big)^3 |U_0|^2 \frac{d\zeta}{1 + |\zeta|} \\ &\leqslant C \big\|(u_0, v_0)\big\|_{H^1}^2, \end{aligned}$$

and we simply have to replace it by

$$\begin{aligned} \int\limits_R^\infty \big(1 + \big|\lambda_\pm(\zeta)\big|\big)^{m+(1/2)} \big|B_\pm(0,\zeta)U_0\big|^2 \frac{d\zeta}{|\lambda'_\pm(\zeta)|} &\leqslant C \int\limits_R^\infty \big(1 + |\zeta|\big)^{2m+1} |U_0|^2 \frac{d\zeta}{1 + |\zeta|} \\ &\leqslant C \big\|(u_0, v_0)\big\|_{H^m}^2. \end{aligned}$$

• Pure boundary value problem: the compatibility conditions ensure that the data of the pure boundary value problems are flat at $t = 0$. The first step is to obtain time-like regularity. We start with the estimate (44):

$$\frac{\alpha\gamma}{2\sqrt{|\tau|}}\|U\|^2 + \beta\big|U(0)\big|^2 \leqslant C|\tau||\varphi|^2,$$

and multiply it by $|\tau|^{m+1/2}$ (instead of $\sqrt{|\tau|}$ as it was done for the low regularity theorem), this gives as for (47):

$$\frac{\gamma}{2}\big\|(u,v)\big\|^2_{L^2_x(H^{m/2}_{\gamma,t})} + \frac{\gamma}{2}\big\|\partial_x(u,v)\big\|_{L^2_x(H^{(m-1)/2})} + \beta\big\|(u,v)(0)\big\|_{H^{m/2+1/4}_{\gamma,t}} + \beta\big\|\partial_x(u,v)(0)\big\|_{H^{(m-1)/2+1/4}_{\gamma,t}} \leqslant C\|\varphi\|^2_{H^{m/2+1/4}_{\gamma,t}}.$$

If we differentiate $\partial_x U = \sqrt{|\tau|}HU$ with respect to x, this gives a new estimate

$$\frac{\alpha\gamma}{2\sqrt{|\tau|}}\|\partial_x U\|^2 + \beta\big|\partial_x U(0)\big|^2 \leqslant C|\tau|\big|\big(\partial_x\widehat{u}(0), \partial_x\widehat{v}(0)\big)\big|^2,$$

and with the estimate (44) we have

$$\frac{\alpha\gamma}{2\sqrt{|\tau|}}\|\partial_x U\|^2 + \beta\big|\partial_x U(0)\big|^2 \leqslant C|\tau|^2|\varphi|^2.$$

Proceeding inductively finally gives

$$\frac{\gamma}{2}\big\|(u,v)\big\|^2_{H^{m,2}_\gamma(\mathbb{R}^+\times\mathbb{R})} + \beta\big\|(u,v)(0)\big\|_{H^{m/2+1/4}_\gamma} + \beta\big\|\partial_x(u,v)(0)\big\|_{H^{(m-1)/2+1/4}_\gamma} \leqslant C'\|\varphi\|^2_{H^{m/2+1/4}_\gamma}. \qquad \square$$

5. Conclusion

The construction of Kreiss symmetrizers for the extended system has allowed us to obtain a priori estimates and well-posedness results. It would be of very high interest to know how far the hyperbolic (homogeneous) theory may be adapted to the fully nonlinear, dispersive problem. A first step would be to use a quasi-homogeneous pseudo/para-differential calculus to treat the case of low regularity variable coefficients (as was initiated by Majda [9]), and eventually obtain at least local or for small data well-posedness results as was done by Mokrane [12] or Métivier [10]. The formal algebraic construction is actually expected to be a straightforward adaptation of what was done in this article. It is the passage from a Kreiss symmetrizer to a functional symmetrizer that involves very technical difficulties that do not arise in the case of hyperbolic systems.

The investigation of Kreiss symmetrizers for more general dispersive boundary value problems, which should help to understand more accurately the specific Euler–Korteweg BVP, is the purpose of a work in progress.

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